NORTH-HOLLAND

Linear Preservers of Immanants on Symmetric Matrices

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ABSTRACT

Let $\mathbb{F}$ be an arbitrary subfield of the complex numbers, and let $H_n(\mathbb{F})$ be the space of the $n \times n$ symmetric matrices with entries in $\mathbb{F}$. We describe the linear operators of $H_n(\mathbb{F})$ that preserve an immanant d_χ, where χ is an irreducible nonlinear character of S_n. © Elsevier Science Inc., 1997

1. INTRODUCTION

Let $\mathbb{F}$ be an arbitrary subfield of the complex numbers, $\mathscr{M}_n(\mathbb{F})$ be the linear space of the n-square matrices with elements in $\mathbb{F}$, and $H_n(\mathbb{F})$ be the subspace of $\mathscr{M}_n(\mathbb{F})$ consisting of all the symmetric matrices. Let S_n be the symmetric group of degree n, and χ be an irreducible character of S_n. The function $d_\chi : \mu_n(\mathbb{C}) \to \mathbb{C}$ defined by

$$d_\chi(A) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

for any n-by-n complex matrix $A = (a_{ij})$ is called an immanant. For example, if χ is the alternating character, then d_χ is the determinant, and if $\chi = 1$, then d_χ is called the permanent.

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In [2] are characterized the linear transformations of $\mathcal{M}_n(\mathbb{F})$ into itself that preserve an immanant d_χ, where χ is an irreducible nonlinear character of S_n. In [3] are studied the linear transformations T on the space of the n-square symmetric matrices over any subfield of the real field that preserve the permanent.

In this paper we characterize the linear transformations on $H_n(\mathbb{F})$ which preserve an immanant d_χ, when χ is a nonlinear character of S_n.

2. MAIN RESULTS

A linear transformation $T : H_n(\mathbb{F}) \to H_n(\mathbb{F})$ preserves an immanant d_χ if

$$d_\chi(T(X)) = d_\chi(X) \qquad \text{for all} \quad X \in H_n(\mathbb{F}).$$

In the first theorem we prove that the immanant preservers of the symmetric matrices must be nonsingular.

THEOREM 2.1. *Let χ be an irreducible nonlinear complex character of S_n, and $\mathbb{F}$ be a subfield of $\mathbb{C}$. If a linear transformation $T : H_n(\mathbb{F}) \to H_n(\mathbb{F})$ preserves the immanant d_χ, then T is nonsingular.*

If $C \in \mathcal{M}_n(\mathbb{F})$ and $X \in \mathcal{M}_n(\mathbb{F})$, we define the Hadamard product of C and X as the matrix $Y = C * X \in \mathcal{M}_n(\mathbb{F})$ where $y_{ij} = c_{ij}x_{ij}$ for all $i, j \in \{1, \dots, n\}$.

If $\sigma \in S_n$, we denote by $P(\sigma)$ the $n \times n$ permutation matrix whose (i, j) entry is $P(\sigma)_{ij} = \delta_{i\sigma(j)}$ for all $i, j \in \{1, \dots, n\}$.

There is a natural one-to-one correspondence between the irreducible characters of S_n and the nonincreasing partitions of n, which are sequences $(m_1, m_2, \dots, m_t)$ where $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_t \geqslant 1$ and $m_1 + m_2 + \cdots + m_t = n$. The character χ associated to the partition $(m_1, m_2, \dots, m_t)$ will be denoted by $\chi = [m_1, m_2, \dots, m_t]$.

THEOREM 2.2. *Let $n > 3$, χ be an irreducible nonlinear complex character of S_n, and $\mathbb{F}$ be a subfield of $\mathbb{C}$. A linear transformation*

$$T : H_n(\mathbb{F}) \to H_n(\mathbb{F})$$

preserves the immanant d_χ if and only if there are a permutation $\sigma \in S_n$ and a matrix $C \in H_n(\mathbb{F})$ such that

$$T(X) = C * P(\sigma)XP(\sigma^{-1}) \qquad \textit{for all} \quad X \in H_n(\mathbb{F}) \tag{2.1}$$

and

$$\prod_{t=1}^{n} c_{t\pi(t)} = 1 \qquad \text{for all} \quad \pi \in S_n \text{ such that } \chi(\pi) \neq 0. \tag{2.2}$$

THEOREM 2.3. *Let* $\mathbb{F}$ *be a subfield of* $\mathbb{C}$, *and* χ *be the character* [2, 1] *of* S_3. *A linear transformation* $T : H_3(\mathbb{F}) \to H_3(\mathbb{F})$ *preserves* d_χ *if and only if there are permutations* $\sigma, \rho \in S_3$ *and a matrix* $C \in H_3(\mathbb{F})$ *satisfying one of the following conditions*:

(a) *one has*

$$T(X) = C * \begin{bmatrix} x_{\sigma(1)} & y_{\rho(1)} & y_{\rho(2)} \\ y_{\rho(1)} & x_{\sigma(2)} & y_{\rho(3)} \\ y_{\rho(2)} & y_{\rho(3)} & x_{\sigma(3)} \end{bmatrix}$$

$$\text{for all} \quad X = \begin{bmatrix} x_1 & y_1 & y_2 \\ y_1 & x_2 & y_3 \\ y_2 & y_3 & x_3 \end{bmatrix} \in H_3(\mathbb{F}) \tag{2.3}$$

and

$$c_{11}c_{22}c_{33} = c_{12}c_{23}c_{13} = 1, \tag{2.4}$$

or

(b) *one has*

$$T(X) = C * \begin{bmatrix} y_{\rho(1)} & x_{\sigma(1)} & x_{\sigma(2)} \\ x_{\sigma(1)} & y_{\rho(2)} & x_{\sigma(3)} \\ x_{\sigma(2)} & x_{\sigma(3)} & y_{\rho(3)} \end{bmatrix}$$

$$\text{for all} \quad X = \begin{bmatrix} x_1 & y_1 & y_2 \\ y_1 & x_2 & y_3 \\ y_2 & y_3 & x_3 \end{bmatrix} \in H_3(\mathbb{F}) \tag{2.5}$$

and

$$c_{11}c_{22}c_{33} = c_{12}c_{23}c_{13} = -1. \tag{2.6}$$

3. PRELIMINARIES

Let us consider some remarks that are useful for the proofs of the main results.

Denote by supp τ the set $\{i_1, \dots, i_s\}$, where $\tau = (i_1 \dots i_s)$ is a cycle of S_n.

REMARK 3.1. Let $\pi, \rho \in S_n$, and τ be a cycle contained in the cycle decomposition of π. Then

$$\rho(t) = \pi(t) \text{ or } \rho(t) = \pi^{-1}(t) \qquad \text{for all} \quad t \in \operatorname{supp} \tau$$

if and only if one of the following conditions holds:

(a) τ or τ^{-1} is a cycle in the cycle decomposition of ρ;

(b) τ has even length, say $\tau = (i_1 i_2 \dots i_{2r})$, and the cycle decomposition of ρ contains one of the following sets of transpositions: $\{(i_1 i_2), \dots, (i_{2r-1} i_{2r})\}$ or $\{(i_1 i_{2r}), (i_{2r-1} i_{2r-2}), \dots, (i_3 i_2)\}$.

Let $\tau_1 \dots \tau_k$ be the cycle decomposition of a permutation $\pi \in S_n$. Denote by C_π the set of permutations of the form $\tau_1^{p_1} \dots \tau_k^{p_k}$ with $p_i \in \{-1, 1\}$ for all $i = 1, \dots, k$, and by D_π the set of permutations $\rho \in S_n$ such that $\rho(t) = \pi(t)$ or $\rho(t) = \pi^{-1}(t)$ for all $t \in \{1, \dots, n\}$.

From Remark 3.1, we can conclude the following remark.

REMARK 3.2.

(a) $C_\pi \subseteq D_\pi$; and $C_\pi = D_\pi$ if and only if π does not contain in its cycle decomposition any cycle of even length greater than two.

(b) If $\rho \in D_\pi - C_\pi$, then ρ contains in its cycle decomposition more transpositions and less cycles of even length greater than two than π does.

The sets C_π will play an important role in the study of the immanant preservers defined on symmetric matrices. Note that all permutations of C_π have the same cycle structure as π. So C_π is contained in the conjugacy class of π.

REMARK 3.3. Let χ be a character of S_n, $\pi \in S_n$, and $\rho \in C_\pi$. Then

$$\chi(\rho) \prod_{t=1}^{n} x_{t\rho(t)} = \chi(\pi) \prod_{t=1}^{n} x_{t\pi(t)} \qquad \text{for all} \quad X \in H_n(\mathbb{F}).$$

An irreducible character χ of S_n is a triangular character if

$$\chi = [m, m-1, \dots, 1],$$

where $m \geqslant 1$.

Triangular characters vanish on all conjugacy classes whose cycle decomposition contains a cycle of even length—in particular, when they contain at least one transposition.

Consider now the following propositions, which are immediate consequences of the main results of [1], and which are also useful for the proofs of the main theorems.

PROPOSITION 3.4. *Let χ be an irreducible nontriangular character of S_n. If i, j are distinct elements of $\{1, \dots, n\}$ there is a permutation ρ that fixes i and j and is such that $\chi(\rho(ij)) \neq 0$.*

PROPOSITION 3.5. *Let χ be an irreducible nontriangular character of S_n and i, j be distinct elements of $\{1, \dots, n\}$. There exists a permutation ρ that fixes i and j and satisfies one of the following conditions:*

(i) $|\chi(\rho(ij))| = |\chi(\rho)| \neq 0$.
(ii) $\chi(\rho(ij)) \neq 0$ *and* $\chi(\rho) = 0$.
(iii) $\chi(\rho(ij)) = 0$ *and* $\chi(\rho) \neq 0$.

PROPOSITION 3.6. *Let $n > 2$ and χ be an irreducible nontriangular character of S_n. If p, q, r are distinct elements of $\{1, \dots, n\}$ and $\chi \neq [2,2]$, there is a permutation ρ that fixes p, q, and r and such that $\chi(\rho(pq)) \neq 0$.*

4. PROOFS

We begin this section with the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $A = (a_{ij}) \in H_n(\mathbb{F})$, be a matrix such that $T(A) = 0$. Then

$$d_\chi(\alpha A + B) = d_\chi(T(\alpha A + B)) = d_\chi(T(B)) = d_\chi(B) \quad \text{for all} \quad \alpha \in \mathbb{F} \text{ and } B \in H_n(\mathbb{F}). \tag{4.1}$$

Let x be an indeterminate over $\mathbb{F}$. Since $\mathbb{F}$ is infinite, we conclude by (4.1) that for any symmetric matrix B with entries in $\mathbb{F}[x]$, the polynomials $d_\chi(xA + B)$ and $d_\chi(B)$ must be equal. With suitable choices of B we will get $a_{ij} = 0$ for all $i, j \in \{1, \dots, n\}$.

Let us fix i and choose B defined by

$$\begin{aligned} b_{ii} &= 0, \\ b_{kk} &= 1 - xa_{kk} \qquad \text{if} \quad k \neq i, \\ b_{pq} &= -xa_{pq} \qquad \text{if} \quad p \neq q, \end{aligned}$$

and let us compute $d_\chi(xA + B)$. We have $d_\chi(xA + B) = a_{ii}\,\chi(\mathrm{id})x$, which must be equal to $d_\chi(B)$. Since $d_\chi(B)$ is a polynomial in x without terms of degree 1, we conclude that $a_{ii} = 0$.

To prove that $a_{ij} = 0$ for all distinct i and j, let us consider the two cases, when χ is a triangular character, and when χ is not a triangular character.

Assume first that χ is a nontriangular character. Let us fix distinct i and j. Using Proposition 3.4, there is a permutation containing the transposition (ij) in its cycle decomposition and where χ does not vanish. Let σ be in these conditions, but containing in its cycle decomposition the maximum number of transpositions.

Consider the matrix B defined by

$$\begin{aligned} b_{ij} &= 0 = b_{ji}, \\ b_{k\sigma(k)} &= 1 - xa_{k\sigma(k)} = b_{\sigma(k)k} \qquad \text{if} \quad k \neq i, j, \\ b_{pq} &= -xa_{pq} \qquad \text{for all the other entries of } B. \end{aligned}$$

Let us compute $d_\chi(xA + B)$. Suppose $\pi \in S_n$ satisfies $\prod_{t=1}^{n}(xA + B)_{t\pi(t)} \neq 0$. Then it is obvious that $\pi(t) = \sigma(t)$ or $\pi(t) = \sigma^{-1}(t)$ for all $t \in \{1, \dots, n\}$, that is, $\pi \in D_\sigma$. Using Remark 3.2, we conclude that $\pi \in C_\sigma$ or π contains in its cycle decomposition more transpositions than σ. By definition of σ this second condition happens only if $\chi(\pi) = 0$. Thus using Remark 3.3, we get

$$\begin{aligned} d_\chi(xA + B) &= \sum_{\pi \in C_\sigma} \chi(\pi) \prod_{t=1}^{n} (xA + B)_{t\pi(t)} \\ &= |C_\sigma|\,\chi(\sigma) \prod_{t=1}^{n} (xA + B)_{t\sigma(t)} \\ &= |C_\sigma|\,\chi(\sigma)(a_{ij})^2 x^2. \end{aligned}$$

Let us study now the polynomial $d_\chi(B)$. Suppose $\pi \in S_n$ satisfies $\prod_{t=1}^n b_{t\pi(t)} \neq 0$. Then $\pi(i) \neq j$ and $\pi(j) \neq i$, and thus we have $b_{i\pi(i)} = -xa_{i\pi(i)}$ and $b_{j\pi(j)} = -xa_{j\pi(j)}$. So if there is $k \neq i, j$ such that $\pi(k) \neq \sigma(k)$ and $\pi(k) \neq \sigma^{-1}(k)$, we will have $b_{k\pi(k)} = -xa_{k\pi(k)}$, and $\prod_{t=1}^n b_{t\pi(t)}$ will not contain a monomial of degree 2. If $\pi(k) = \sigma(k)$ or $\pi(k) = \sigma^{-1}(k)$ for all $k \neq i, j$, using Remark 3.1, we get $\pi(i) = i$ and thus $b_{i\pi(i)} = b_{ii} = -xa_{ii} = 0$, which is a contradiction.

Therefore the polynomial $d_\chi(B)$ does not contain any monomial of degree 2. Since $d_\chi(B) = d_\chi(xA + B) = |C_\sigma|\,\chi(\sigma)(a_{ij})^2x^2$, we must have $a_{ij} = 0$.

Assume now that χ is a triangular character. Let us fix distinct i and j. Let $k \neq i, j$ and $\sigma = (ijk)$, and define the matrix B by

$$\begin{aligned} b_{ij} &= 0 = b_{ji}, \\ b_{t\sigma(t)} &= 1 - xa_{t\sigma(t)} = b_{\sigma(t)t} && \text{if} \quad t \neq i, \\ b_{pq} &= -xa_{pq} && \text{for all the other entries of } B. \end{aligned}$$

If $\prod_{t=1}^n (xA + B)_{t\pi(t)} \neq 0$ for some π, then we can easily conclude that $\pi = \sigma$ or $\pi = \sigma^{-1}$. So we have $d_\chi(xA + B) = 2\chi(\sigma)\prod_{t=1}^n (xA + B)_{t\sigma(t)} = 2\chi(\sigma)a_{ij}x$.

Assume now that for some π we have $\prod_{t=1}^n b_{t\pi(t)} \neq 0$, and that $\prod_{t=1}^n b_{t\pi(t)}$ contains a monomial of degree one. Then we must have $\pi = (ik)$ or $\pi = (jk)$ and consequently $\chi(\pi) = 0$, since χ vanishes on the transpositions. Thus the polynomial $d_\chi(B)$ does not contain monomials of degree 1. On the other hand, by Proposition 2.4 of [1], the character χ does not vanish on the cycles of length 3, so $\chi(\sigma) \neq 0$. Therefore we must have $a_{ij} = 0$. ∎

We will denote by U_{ij} the $n \times n$ matrix with 1 in positions (i, j) and (j, i) and 0 elsewhere. It is well known that the family $\{U_{ij} : i \leqslant j,\ i, j = 1, \ldots, n\}$ is a basis of $H_n(\mathbb{F})$ over $\mathbb{F}$.

In what follows χ is a fixed irreducible nonlinear character of S_n, $\mathbb{F}$ is a subfield of $\mathbb{C}$, and $T : H_n(\mathbb{F}) \to H_n(\mathbb{F})$ is a linear preserver of the immanant d_χ. Let $\mathscr{A}'$ and $\mathscr{A}''$ be the subsets of $H_n(\mathbb{F})$ defined by

$$\mathscr{A}' = \{A : \deg d_\chi(xA + B) \leqslant 1 \text{ for all } B \in H_n(\mathbb{F})\},$$

$$\mathscr{A}'' = \{A : \deg d_\chi(xA + B) \leqslant 2 \text{ for all } B \in H_n(\mathbb{F})\},$$

where $\deg p(x)$ denotes the degree of the polynomial $p(x)$.

The proof of Theorem 2.2 is divided into several lemmas. In a first step we study the sets $\mathscr{A}'$ and $\mathscr{A}''$. Using Theorem 2.1, we can easily conclude that $\mathscr{A}'$ and $\mathscr{A}''$ are invariant for T, that is $T(\mathscr{A}') \subseteq \mathscr{A}'$ and $T(\mathscr{A}'') \subseteq \mathscr{A}''$. Then we characterize the image of some particular matrices of $\mathscr{A}'$ and $\mathscr{A}''$. It is necessary to consider separately the cases when the character χ is a triangular character and when it is not.

LEMMA 4.1. *Let $A \in \mathscr{A}'$ and i, j be distinct elements of $\{1, \ldots, n\}$. If $\rho \in S_n$ fixes i and j, then*

$$\chi(\rho) a_{ii} a_{jj} + \chi(\rho(ij))(a_{ij})^2 = 0.$$

Proof. Let ρ be a permutation that fixes i and j, and let $\sigma = \rho(ij)$. Take as B the matrix defined by

$$b_{k\sigma(k)} = b_{\sigma(k)k} = 1 \qquad \text{if} \quad k \neq i, j$$

and

$$b_{pq} = 0 \qquad \text{in the remaining cases},$$

and remark that since $A \in \mathscr{A}'$, the coefficient of x^2 in $d_\chi(xA + B)$ must be zero. But it is equal to

$$\sum_{\pi \in S} \chi(\pi) a_{i\pi(i)} a_{j\pi(j)},$$

where S is the set of permutations π such that $\pi(k) = \sigma(k)$ or $\pi(k) = \sigma^{-1}(k)$ for all $k \neq i, j$. Using the remarks of the previous section, it is clear that $S = D_\rho \cup D_\sigma$ and $D_\sigma = \{\pi(ij) : \pi \in D_\rho\}$. Thus we get

$$\sum_{\pi \in D_\rho} \left[\chi(\pi) a_{ii} a_{jj} + \chi(\pi(ij))(a_{ij})^2 \right] = 0.$$

Now if ρ does not contain in its cyclic decomposition any cycle of even length greater than two, then $C_\rho = D_\rho$ and the above equality becomes

$$|C_\rho| \left[\chi(\rho) a_{ii} a_{jj} + \chi(\rho(ij))(a_{ij})^2 \right] = 0.$$

The result follows easily by induction on the number of cycles of even length greater than two in the cycle decomposition of ρ, and using the remarks of Section 3. ∎

LEMMA 4.2. *Let $n > 3$, and χ be a triangular character. Then $\mathscr{A}$ consists of the set of matrices*

$$\{\alpha U_{ii} + \beta U_{ik} : i \neq k,\ i, k = 1, \dots, n, \text{ and } \alpha, \beta \in \mathbb{F}\}.$$

Proof. Let $A = \alpha U_{ii} + \beta U_{ik}$, where i and k are distinct elements belonging to the set $\{1, \dots, n\}$, and $\alpha, \beta \in \mathbb{F}$. If $\pi \in S_n$ and the transposition (ik) does not occur in the cycle decomposition of π, then

$$\deg\left[\prod_{t=1}^{n} (xA + B)_{t\pi(t)}\right] \leqslant 1$$

for all $B \in H_n(\mathbb{F})$. If the transposition (ik) belongs to the cycle decomposition of π, as χ is a triangular character, we have that $\chi(\pi) = 0$. So $\deg[d_\chi(xA + B)] \leqslant 1$, for all $B \in H_n(\mathbb{F})$, that is, $A \in \mathscr{A}$.

The converse follows easily from the following steps:

Step 1. If $A \in \mathscr{A}$ and i, j are distinct elements of $\{1, \dots, n\}$, then $a_{ii}a_{jj} = 0$.
Step 2. If $A \in \mathscr{A}$ and i, p, h are distinct elements of $\{1, \dots, n\}$, then $a_{ii}a_{ph} = 0$.
Step 3. If $A \in \mathscr{A}$ and i, h, p are distinct elements of $\{1, \dots, n\}$, then $a_{ih}a_{pi} = 0$.
Step 4. If $A \in \mathscr{A}$ and i, h, p, r are distinct elements of $\{1, \dots, n\}$, then $a_{ih}a_{pr} = 0$.

Let us now prove these steps:

Step 1: This is an immediate consequence of considering $\rho = \mathrm{id}$ in Lemma 4.1 and the fact that a triangular character vanishes on the transpositions.

Step 2: Let $\sigma = (phk)$, where k is distinct from i, p, h. By Proposition 2.4 of [1], we have that $\chi(\sigma) \neq 0$. Let $B \in H_n(\mathbb{F})$ be defined by

$$b_{t\sigma(t)} = b_{\sigma(t)t} = 1 \qquad \text{if} \quad t \neq i, p,$$
$$b_{rs} = 0 \qquad \text{in the remaining cases.}$$

Assume π is a permutation such that the polynomial $\prod_{t=1}^{n}(xA + B)_{t\pi(t)}$ contains a monomial of degree 2. Since $b_{i\pi(i)} = 0$, then we must have $\pi(h) = k$ or $\pi(p) = k$, and clearly, $\pi \in \{\sigma, \sigma^{-1}, (kh), (kp), (kh)(ip), (kp)(ih), (hkpi), (pkhi)\}$. Since χ is a triangular character, χ vanishes on all these permutations, except on σ and on σ^{-1}. Then the coefficient of x^2 in the polynomial $d_\chi(xA + B)$ is $2\chi(\sigma)a_{ii}a_{ph}$; thus $a_{ii}a_{ph} = 0$.

Step 3: Let $\sigma = (ihp)$, define $B \in H_n(\mathbb{F})$ as in step 2, and assume that π is a permutation such that the polynomial $\prod_{t=1}^{n}(xA + B)_{t\pi(t)}$ contains a monomial of degree 2. Then $\pi = \sigma$ or $\pi = \sigma^{-1}$ or the transposition (hp) belongs to the cycle decomposition of π. Since the character χ is triangular, the coefficient of x^2 in the polynomial $d_\chi(xA + B)$ is $2\chi(\sigma)a_{ih}a_{pi}$, and the result follows, since $\chi(\sigma) \neq 0$.

Step 4: Assume that $a_{ih} \neq 0$ and that $a_{pr} \neq 0$. Applying step 2, we conclude that all the principal elements of A are zero, and then, using step 3, we prove that a_{ih} is the only nonzero entry of row i and the only nonzero entry of column h, and that a_{pr} is the only nonzero entry of row p and the only nonzero entry of column r.

Since χ is triangular and $n > 3$, we must have $n \geqslant 6$, and using Proposition 2.4 of [1], we know that χ does not vanish on the cycles of length 5. Let $\sigma = (ihprq)$, where $q \notin \{i, h, p, r\}$, and define $B \in H_n(\mathbb{F})$ by

$$\begin{aligned} b_{t\sigma(t)} = b_{\sigma(t)t} &= 1 \qquad \text{if} \quad t \neq i, p, \\ b_{js} &= 0 \qquad \text{in the remaining cases.} \end{aligned}$$

Assume that π is a permutation such that $\chi(\pi) \neq 0$, $\prod_{t=1}^{n}(xA + B)_{t\pi(t)} \neq 0$, and this polynomial contains a monomial of degree 2. Let us examine what the permutation π may be. If $\pi(i) \neq h$, then $a_{i\pi(i)} = 0$, and if $\pi(i) \neq q$, then $b_{i\pi(i)} = 0$, so we must have

$$\pi(i) = h = \sigma(i) \quad \text{or} \quad \pi(i) = q = \sigma^{-1}(i).$$

With similar arguments we prove that

$$\pi(h) = p = \sigma(h) \quad \text{or} \quad \pi(h) = i = \sigma^{-1}(h),$$

$$\pi(p) = r = \sigma(p) \quad \text{or} \quad \pi(p) = h = \sigma^{-1}(p),$$

and

$$\pi(r) = q = \sigma(r) \quad \text{or} \quad \pi(r) = p = \sigma^{-1}(r).$$

Since $\chi(\pi) \neq 0$, the cycle decomposition of π does not contain any transposition. So if $\pi(i) = h$, then $\pi(h) = p$, $\pi(p) = r$, $\pi(r) = q$. Under these conditions we must have $b_{t\pi(t)} = 1$ whenever $t \notin \operatorname{supp} \sigma$, because $b_{ih} = b_{pr} = 0$. Consequently $\pi(t) = t$ if $t \notin \operatorname{supp} \sigma$. Then $\pi(q) = i$ and $\pi = \sigma$.

If $\pi(i) = q$, then $\pi(r) = p$, $\pi(p) = h$, $\pi(h) = i$. Since $b_{rp} = b_{hi} = 0$, using the same arguments as before, we prove that $\pi = \sigma^{-1}$.

Therefore the coefficient of x^2 in $d_\chi(xA + B)$ is $2\chi(\sigma)a_{ih}a_{pr} \neq 0$, and we come to a contradiction. ∎

LEMMA 4.3. *If χ is an irreducible nontriangular character of* S_n, *then* $\mathscr{A}'$ *consists of the set of matrices*

$$\{\alpha U_{ii} : i = 1, \dots, n \text{ and } \alpha \in \mathbb{F}\}.$$

Proof. It is clear that $\alpha U_{ii} \in \mathscr{A}'$ for all $i \in \{1, \dots, n\}$ and $\alpha \in \mathbb{F}$.

Conversely, assume that $A \in \mathscr{A}'$, and let i, j be distinct elements of $\{1, \dots, n\}$. Take a permutation ρ, fixing i and j and satisfying one of the conditions of Proposition 3.5. If ρ satisfies condition (ii), by Lemma 4.1 we conclude that $a_{ij} = 0$. Assume that ρ satisfies (i). By Lemma 4.1 we obtain the equalities

$$\chi(\mathrm{id}) a_{ii} a_{jj} + \chi(ij)(a_{ij})^2 = 0, \tag{4.2}$$

$$\chi(\rho) a_{ii} a_{jj} + \chi(\rho(ij))(a_{ij})^2 = 0. \tag{4.3}$$

Multiplying (4.2) by $\chi(\rho(ij))$ and (4.3) by $-\chi(ij)$, and adding the results, we obtain the equality

$$(\chi(\mathrm{id})\chi(\rho(ij)) - \chi(\rho)\chi(ij)) a_{ii} a_{jj} = 0.$$

Therefore $a_{ii}a_{jj} = 0$, since if not we obtain $|\chi(ij)| = \chi(\mathrm{id})$, which is not true because χ is not a linear character. Since $\chi(\rho(ij)) \neq 0$, using (4.3) once more we get $a_{ij} = 0$.

Finally, suppose that ρ satisfies condition (iii). By Lemma 4.1, we get $a_{ii}a_{jj} = 0$. Take now a permutation ρ' that fixes i and j and such that $\chi(\rho'(ij)) \neq 0$. Applying once more Lemma 4.1 and the fact that $a_{ii}a_{jj} = 0$, we must have $a_{ij} = 0$.

So A is a diagonal matrix, and if in Lemma 4.1 we consider $\rho = \mathrm{id}$, we conclude that A has at most one principal element that is different from zero, and the result follows. ∎

LEMMA 4.4. *Let* $A \in \mathscr{A}''$. *If* p, q, r *are distinct elements of* $\{1, \dots, n\}$, *and* $\rho \in S_n$ *fixes* p, q, *and* r, *then*

$$\chi(\rho) a_{pp} a_{qq} a_{rr} + \chi(\rho(pq))\left[a_{pp}(a_{qr})^2 + a_{qq}(a_{pr})^2 + a_{rr}(a_{pq})^2\right]$$
$$+ 2\chi(\rho(pqr)) a_{pq} a_{qr} a_{pr} = 0.$$

Proof. The proof is similar to that of Lemma 4.1. ∎

In what follows it is important to remark that $T(\mathscr{A}') \subseteq \mathscr{A}'$, $T(\mathscr{A}'') \subseteq \mathscr{A}''$, and T is nonsingular. Let us denote by $h(X)$ the number of nonzero elements of a matrix X.

LEMMA 4.5. *If $n > 3$, there is a permutation $\sigma \in S_n$ such that*

$$T(U_{ij}) \in \langle U_{\sigma(i)\sigma(j)} \rangle \qquad \textit{for all} \quad i, j \in \{1, \dots, n\}.$$

Proof. First let us prove the following

CLAIM. *For all $i \in \{1, \dots, n\}$ there is $s \in \{1, \dots, n\}$ such that $T(U_{ii}) \in \langle U_{ss} \rangle$.*

If χ is a nontriangular character, the result follows on applying Lemma 4.3 and the fact that $T(\mathscr{A}') \subseteq \mathscr{A}'$.

Assume now that χ is triangular. By Lemma 4.2, if $A \in \mathscr{A}'$, then $h(A) \leqslant 3$ and A has at most one principal element that is different from zero.

Let us fix $i \in \{1, \dots, n\}$. Since $T(\mathscr{A}') \subseteq \mathscr{A}'$, by Lemma 4.2 we get

$$h(T(U_{ii})) \in \{1, 2, 3\}$$

and

$$h\big(T(U_{ii}) + T(U_{ij})\big) \in \{1, 2, 3\} \qquad \text{for all } j \text{ distinct from } i, \tag{4.4}$$

$$h\big(T(U_{ij})\big) \in \{1, 2, 3\} \qquad \text{for all } j \text{ distinct from } i. \tag{4.5}$$

Assume now that $h(T(U_{ii})) = 3$, that is, there are nonzero elements $\alpha, \beta \in \mathbb{F}$ and distinct p and q such that $T(U_{ii}) = \alpha U_{pp} + \beta U_{pq}$. Using (4.4) and (4.5), we conclude that for all j distinct from i, there are α_j, $\beta_j \in \mathbb{F}$ such that $T(U_{ij}) = \alpha_j U_{pp} + \beta_j U_{pq}$. Therefore the matrices $T(U_{i1}), \dots, T(U_{in})$ belong to the subspace $\langle U_{pp}, U_{pq} \rangle$, which cannot happen, since T is nonsingular, the subspace $\langle U_{i1}, \dots, U_{in} \rangle$ has dimension n, and $n > 2$.

Suppose that $h(T(U_{ii})) = 2$, that is, there are $\alpha \in \mathbb{F}$, $\alpha \neq 0$, and distinct p and q such that $T(U_{ii}) = \alpha U_{pq}$. Arguing as before, we have that for all j distinct from i, $T(U_{ij}) \in \langle U_{pp}, U_{pq} \rangle$ or $T(U_{ij}) \in \langle U_{qq}, U_{pq} \rangle$; therefore the matrices $T(U_{i1}), \dots, T(U_{in}) \in \langle U_{pp}, U_{qq}, U_{pq} \rangle$, and once more we come to a contradiction.

So we must have $h(T(U_{ii})) = 1$ for all $i \in \{1, \dots, n\}$, which proves our claim.

Now as T is nonsingular, there must be a permutation $\sigma \in S_n$ such that $T(U_{ii}) \in \langle U_{\sigma(i)\sigma(i)} \rangle$ for all $i \in \{1, \dots, n\}$, and to get the result it is enough to prove that if $i \neq j$, then $T(U_{ij}) \in \langle U_{\sigma(i)\sigma(j)} \rangle$. We will consider separately the following three cases:

Case 1. χ is triangular.
Case 2. χ is not triangular and $\chi \neq [2,2]$.
Case 3. $\chi = [2,2]$.

Case 1: Let $i \neq j$. If $T(U_{ij})$ has a principal element which is different from zero, then one of the matrices $T(U_{ii}) + T(U_{ij})$ or $T(U_{jj}) + T(U_{ij})$ has two nonzero principal elements, which contradicts Lemma 4.2, since these matrices belong to $\mathscr{A}'$. The result follows on attending to the fact that $T(U_{ij}) \in \mathscr{A}'$, and using once more Lemma 4.2.

Case 2: Let us fix $i \neq j$ and define $p = \sigma(i)$, $q = \sigma(j)$, and $A = T(U_{ij})$. Since $U_{ij} + aU_{ii} + bU_{jj} \in \mathscr{A}''$ for all $a, b \in \mathbb{F}$, and $T(\mathscr{A}'') \subseteq \mathscr{A}''$, then $A + \alpha U_{pp} + \beta U_{qq} \in \mathscr{A}''$ for all $\alpha, \beta \in \mathbb{F}$.

Let r be different from p and q, and let ρ be a permutation that fixes p, q, r. Applying Lemma 4.4 to the matrix $A + \alpha U_{pp} + \beta U_{qq}$, we conclude that

$$\begin{aligned}\chi(\rho)(a_{pp} + \alpha)(a_{qq} + \beta)a_{rr} &\\ + \chi(\rho(pq))\left[(a_{pp} + \alpha)(a_{qr})^2 + (a_{qq} + \beta)(a_{pr})^2 + a_{rr}(a_{pq})^2\right] &\\ + 2\chi(\rho(pqr))a_{pq}a_{qr}a_{pr} = 0, \qquad \text{for all} \quad \alpha, \beta \in \mathbb{F}. &\qquad (4.6)\end{aligned}$$

Taking $\alpha = -a_{pp}$ and $\beta = -a_{qq}$, we obtain

$$\chi(\rho(pq))a_{rr}(a_{pq})^2 + 2\chi(\rho(pqr))a_{pq}a_{qr}a_{pr} = 0. \qquad (4.7)$$

From (4.6) and (4.7) we get

$$\begin{aligned}\chi(\rho)(a_{pp} + \alpha)(a_{qq} + \beta)a_{rr} &\\ + \chi(\rho(pq))\left[(a_{pp} + \alpha)(a_{qr})^2 + (a_{qq} + \beta)(a_{pr})^2\right] = 0 &\\ \text{for all } \alpha, \beta \in \mathbb{F}. \quad (4.8)\end{aligned}$$

Taking in (4.8) $\alpha = 1 - a_{pp}$ and $\beta = -a_{qq}$, we have that

$$\chi(\rho(pq))(a_{qr})^2 = 0, \qquad (4.9)$$

and taking in (4.8) $\alpha = -a_{pp}$ and $\beta = 1 - a_{qq}$, we get

$$\chi(\rho(pq))(a_{pr})^2 = 0. \tag{4.10}$$

Using (4.9) and (4.10) and taking in (4.8) $\alpha = 1 - a_{pp}$, $\beta = 1 - a_{qq}$, and $\rho = \mathrm{id}$, we obtain $a_{rr} = 0$. Taking in (4.9) and (4.10) a permutation ρ fixing p, q, r and such that $\chi(\rho(pq)) \neq 0$, which exists by Proposition 3.6, we get $a_{qr} = a_{pr} = 0$.

Let h, k be distinct, and both distinct from p and q, and let ρ be a permutation that fixes h, k, and p, and such that $\chi(\rho(hk)) \neq 0$. Applying once more Lemma 4.4, we obtain, for all α,

$$\begin{aligned} &\chi(\rho)a_{hh}a_{kk}(\alpha + a_{pp}) \\ &\quad + \chi(\rho(hk))\left[a_{hh}(a_{kp})^2 + a_{kk}(a_{hp})^2 + (\alpha + a_{pp})(a_{hk})^2\right] \\ &\quad + 2\chi(\rho(hkp))a_{hk}a_{kp}a_{hp} = 0. \end{aligned}$$

Since we have already proved that $a_{hh} = a_{kk} = a_{kp} = a_{hp} = 0$, we get

$$\chi(\rho(hk))(\alpha + a_{pp})(a_{hk})^2 = 0. \tag{4.11}$$

As $\chi(\rho(hk)) \neq 0$, it follows from (4.11), taking $\alpha = 1 - a_{pp}$, that $a_{hk} = 0$. So we have that $A = a_{pp}U_{pp} + a_{qq}U_{qq} + a_{pq}U_{pq}$, and since T is nonsingular, $a_{pq} \neq 0$.

Let us prove now that $a_{pp} = a_{qq} = 0$. Let $k \neq i, j$ and $A' = T(U_{ik})$. As we saw before, if $t = \sigma(k)$, we have

$$A' = a'_{pp}U_{pp} + a'_{tt}U_{tt} + a'_{pt}U_{pt} \quad \text{and} \quad a'_{pt} \neq 0.$$

Since the matrix $aU_{ii} + U_{ij} + U_{ik} \in \mathscr{A}''$ for all $a \in \mathbb{F}$, then $\alpha U_{pp} + A + A' \in \mathscr{A}''$ for all $\alpha \in \mathbb{F}$. Take $\rho = \mathrm{id}$ in Lemma 4.4; then

$$\chi(\mathrm{id})(\alpha + a_{pp} + a'_{pp})a_{qq}a'_{tt} + \chi(pq)\left[a_{qq}(a'_{pt})^2 + a'_{tt}(a_{pq})^2\right] = 0 \quad \text{for all} \quad \alpha \in \mathbb{F}, \tag{4.12}$$

and if we take $\alpha = -a_{pp} - a'_{pp}$, we obtain

$$\chi(pq)\left[a_{qq}(a'_{pt})^2 + a'_{tt}(a_{pq})^2\right] = 0.$$

Substituting this equality in (4.12), we get

$$\chi(\mathrm{id})(\alpha + a_{pp} + a'_{pp})a_{qq}a'_{tt} = 0,$$

and so, if we take $\alpha = 1 - a_{pp} - a'_{pp}$, we conclude that

$$a_{qq}a'_{tt} = 0. \tag{4.13}$$

Define a permutation ρ fixing p, q, t and such that $\chi(\rho(pq)) \neq 0$. Applying Lemma 4.4 to the matrix $A + A'$, we get

$$\chi(\rho(pq))\left[a_{qq}(a'_{pt})^2 + a'_{tt}(a_{pq})^2\right] = 0. \tag{4.14}$$

From (4.13) and (4.14) we obtain $a_{qq} = 0$.

Similarly, since the matrices $\alpha U_{jj} + U_{ij} + U_{jk}$ belong to $\mathscr{A}''$, we prove that $a_{pp} = 0$.

Case 3: Here we have $n = 4$.

FACT 1. *If i and j are distinct, there are distinct r and s such that $T(U_{ij}) \in \langle U_{rs} \rangle$.*

Let i be distinct from j, and $A = T(U_{ij})$. Since the matrix $U_{ij} + \alpha U_{kk} + \beta U_{hh} \in \mathscr{A}''$ for all $\alpha, \beta \in \mathbb{F}$ and all k and h, then $A + \alpha U_{kk} + \beta U_{hh} \in \mathscr{A}''$ for all k and h. Using Lemma 4.4, as before, we obtain

$$a_{rs}a_{st}a_{rt} = 0 \qquad \text{for all pairwise distinct } r, s, \text{ and } t \tag{4.15}$$

and

$$a_{tt} = 0 \qquad \text{for all } t. \tag{4.16}$$

Recall that the character $[2, 2]$ vanishes on the transpositions, and does not vanish on the cycles of length three.

Let $B = U_{rs}$, where $r \neq s$. The coefficient of x^3 in the polynomial $d_\chi(xA + B)$ is equal to $4a_{rs}(a_{kh})^2$, where $\{k, h\} = \{1, 2, 3, 4\} - \{r, s\}$, which gives

$$a_{rs}a_{kh} = 0 \qquad \text{whenever } k, h, r, s \text{ are pairwise distinct.} \tag{4.17}$$

Fix now r and s such that $a_{rs} \neq 0$, and let $\{k, h\} = \{1,2,3,4\} - \{r, s\}$. Then from (4.17) we obtain $a_{kh} = 0$, and from (4.15) $a_{sh}a_{rh} = 0$. So if $a_{rh} \neq 0$, then $a_{sh} = 0$, and (4.17) gives also $a_{sk} = 0$, and attending to the fact that A is symmetric, we have that the nonzero elements of the matrix A belong to row r or to column r. In the same way, if we assume that $a_{sh} \neq 0$, we see that the nonzero elements of the matrix A belong to row s or to column s.

Let us order $\{r, s, k, h\}$ so that $a_{rs} \neq 0$ and the nonzero elements of A belong to the row r or to the column r. Consider the matrix

$$A' = A + (1 - a_{kk})U_{kk} \in \mathscr{A}'',$$

and take $B = U_{rh} + U_{sh}$. The coefficient of x^3 in the polynomial $d_\chi(xA' + B)$ is equal to $-2a_{rs}a_{rh}$. Consequently $a_{rh} = 0$. Using the same arguments as before, we prove that $a_{rk} = 0$, and consequently $A = a_{rs}U_{rs}$, which proves the Fact 1.

Let S be the set of all subsets of cardinal 2. Define the function $g : S \to S$ by $g(\{i,j\}) = \{r, s\}$ if and only if $T(U_{ij}) \in \langle U_{rs}\rangle$. Since T is nonsingular, g is a bijective function. Let us prove the following

FACT 2. *If i, j, k, h are pairwise distinct, then* $g(\{i,j\}) \cup g(\{k,h\}) = \{1,2,3,4\}$.

Consider the matrix $A' = U_{ij} + U_{kh}$. If $g(\{i,j\}) = \{r, s\}$ and $g(\{k,h\}) = \{r', s'\}$, then $\{r, s\} \neq \{r', s'\}$. Then $T(A') = \alpha U_{rs} + \beta U_{r's'}$ has four nonzero elements. But $d_\chi(T(A')) = d_\chi(A') = 2$, and consequently $T(A')$ cannot have zero rows, which gives $\{r, s, r', s'\} = \{1,2,3,4\}$.

Consider now the matrix $B = U_{23} + U_{34} + U_{24} + U_{11}$. Then $d_\chi(T(B)) = d_\chi(B) = -2$. On the other hand, $T(B)$ has only one nonzero principal element, and that is the element in position $(\sigma(1), \sigma(1))$. Attending to Fact 2, we have that if $\{i, j, k, h\} = \{1,2,3,4\}$, then $(T(B))_{ij}(T(B))_{kh} = 0$. Then it is easy to prove that

$$\begin{aligned} d_\chi(T(B)) = {} & -2(T(B))_{\sigma(2)\sigma(3)}(T(B))_{\sigma(3)\sigma(4)} \\ & \times (T(B))_{\sigma(2)\sigma(4)}(T(B))_{\sigma(1)\sigma(1)}. \end{aligned}$$

Then $(T(B))_{\sigma(2)\sigma(3)} \neq 0$, $(T(B))_{\sigma(3)\sigma(4)} \neq 0$, and $(T(B))_{\sigma(2)\sigma(4)} \neq 0$. Since

$$T(B) = \alpha_1 U_{p_1q_1} + \alpha_2 U_{p_2q_2} + \alpha_3 U_{p_3q_3} + \alpha_4 U_{\sigma(1)\sigma(1)}$$

where

$$\{p_1, q_1\} = g(\{2,3\}), \qquad \{p_2, q_2\} = g(\{3,4\}), \qquad \{p_3, q_3\} = g(\{2,4\}),$$

then

$$\{g(\{2,3\}), g(\{3,4\}), g(\{2,4\})\}$$
$$= \{\{\sigma(2), \sigma(3)\}, \{\sigma(3), \sigma(4)\}, \{\sigma(2), \sigma(4)\}\},$$

and consequently

$$\{g(\{1,2\}), g(\{1,3\}), g(\{1,4\})\}$$
$$= \{\{\sigma(1), \sigma(2)\}, \{\sigma(1), \sigma(3)\}, \{\sigma(1), \sigma(4)\}\}.$$

So whenever $j \neq 1$, there is $k \neq \sigma(1)$ such that $T(U_{1j}) \in \langle U_{\sigma(1)k} \rangle$. Similarly we prove that for all i and all $j \neq i$, there is $k \neq \sigma(i)$ such that $T(U_{ij}) \in \langle U_{\sigma(i)k} \rangle$. Since $U_{ij} = U_{ji}$ for all i and j, the result follows. ■

Proof of Theorem 2.2. Let $T : H_n(\mathbb{F}) \to H_n(\mathbb{F})$ be a linear preserver of the immanant d_χ. Using Lemma 4.5 and linearity, we state the existence of a matrix $C \in H_n(\mathbb{F})$ and $\sigma \in S_n$ satisfying the condition (2.1). So it is enough to prove that such a matrix C must satisfy (2.2).

So assume that π is a permutation, such that $\chi(\pi) \neq 0$. Take $\rho = \sigma^{-1}\pi\sigma$ and consider the matrix B defined by $B = \sum_{t=1}^{n} U_{t\rho(t)}$. Then

$$d_\chi(B) = \sum_{\rho' \in D_\rho} \chi(\rho'). \tag{4.18}$$

On the other hand, by (2.1), we get $T(B) = \sum_{t=1}^{n} c_{t\pi(t)} U_{t\pi(t)}$; thus

$$d_\chi(T(B)) = \sum_{\rho' \in D_\pi} \chi(\rho') \prod_{t=1}^{n} c_{t\rho'(t)}. \tag{4.19}$$

Suppose that π and thus ρ do not contain in their cycle decomposition any cycle of even length greater than 2. Then $D_\pi = C_\pi$, $D_\rho = C_\rho$ and the equalities (4.18) and (4.19) become

$$d_\chi(B) = |C_\rho|\, \chi(\rho) \tag{4.20}$$

and

$$d_\chi(T(B)) = |C_\pi|\, \chi(\pi) \prod_{t=1}^{n} c_{t\pi(t)}. \tag{4.21}$$

As ρ and π have the same cyclic structure, we have $|C_\rho| = |C_\pi|$ and $\chi(\rho) = \chi(\pi)$. Since $\chi(\pi) \neq 0$, and T preserves d_χ, from the equalities (4.20) and (4.21) we get $\prod_{t=1}^{n} c_{t\pi(t)} = 1$.

Assume now that π contains in its cycle decomposition k cycles ($k \geqslant 1$) of even length greater than 2, and suppose by induction that $\prod_{t=1}^{n} c_{t\phi(t)} = 1$ whenever $\chi(\phi) \neq 0$ and the number of cycles contained in the decomposition of ϕ that are of even length greater than 2 is less than k. The equalities (4.18) and (4.19) can be written in the following way:

$$d_\chi(B) = |C_\pi| \chi(\pi) + \sum_{\psi \in D_\rho - C_\rho} \chi(\psi) \tag{4.22}$$

and

$$d_\chi(T(B)) = |C_\pi| \chi(\pi) \prod_{t=1}^{n} c_{t\pi(t)} + \sum_{\phi \in D_\pi - C_\pi} \chi(\phi) \prod_{t=1}^{n} c_{t\phi(t)}. \tag{4.23}$$

Since ρ and π have the same cycle structure, it is obvious that there is a bijective map $\theta : D_\pi - C_\pi \to D_\rho - C_\rho$ such that $\theta(\phi)$ and ϕ have the same cycle structure for all $\phi \in D_\pi - C_\pi$. Then from (4.22) we obtain

$$d_\chi(B) = |C_\pi| \chi(\pi) + \sum_{\phi \in D_\pi - C_\pi} \chi(\phi). \tag{4.24}$$

On the other hand, if $\phi \in D_\pi - C_\pi$, then ϕ contains less cycles of even length greater than two than π does, and using induction we must have $\chi(\phi) = 0$ or $\prod_{t=1}^{n} c_{t\phi(t)} = 1$, and in both cases we get $\chi(\phi)\prod_{t=1}^{n} c_{t\phi(t)} = \chi(\phi)$. Thus from (4.23) we obtain

$$d_\chi(T(B)) = |C_\pi| \chi(\pi) \prod_{t=1}^{n} c_{t\pi(t)} + \sum_{\phi \in D_\pi - C_\pi} \chi(\phi). \tag{4.25}$$

Since T preserves d_χ, from (4.24) and (4.25) we obtain $\prod_{t=1}^{n} c_{t\pi(t)} = 1$.

The converse can be easily verified. ∎

Before presenting the proof of Theorem 2.3, let us consider some more lemmas. In what follows $n = 3$, and χ is always the character [2, 1].

LEMMA 4.6. *Let $n = 3$, and χ be the character* [2, 1] *of* S_3. *Then* $\mathscr{A}' = \{\alpha U_{ii} + \beta U_{st} : s \neq t \text{ and } \alpha, \beta \in \mathbb{F}\}$.

Proof. With similar arguments to those used in the proof of Lemma 4.2, we get that the matrices $\alpha U_{ii} + \beta U_{st}$ with $s \neq t$ belong to $\mathscr{A}'$.

The converse is a consequence of steps 1 and 3 of the proof of Lemma 4.2, which do not use the condition $n > 3$. ∎

LEMMA 4.7. *Let χ be the character* [2, 1] *of* S_3, *and T a linear operator of $H_3(\mathbb{F})$ that preserves d_χ. Let $U_1' = U_{12}$, $U_2' = U_{13}$ $U_3' = U_{23}$. Then there are permutations $\pi, \tau \in S_3$ satisfying one of the following conditions*:

(i) $T(U_{ii}) \in \langle U_{\pi(i)\pi(i)} \rangle$ *and* $T(U_i') \in \langle U_{\tau(i)}' \rangle$, $i = 1, 2, 3$, *or*
(ii) $T(U_{ii}) \in \langle U_{\pi(i)}' \rangle$ *and* $T(U_i') \in \langle U_{\tau(i)\tau(i)} \rangle$, $i = 1, 2, 3$.

Proof. Using Lemma 4.6, we have that $h(A) \leqslant 3$ for all $A \in \mathscr{A}'$. Moreover, a matrix of $\mathscr{A}'$ has at most one nonzero principal element.

CLAIM 1. $h(T(U_{ik})) < 3$ *for all i and k.*

Suppose $h(T(U_{ii})) = 3$. Then there are $j, r, p \in \{1, 2, 3\}$ with $r \neq p$ such that $(T(U_{ii}))_{jj} \neq 0$ and $(T(U_{ii}))_{rp} \neq 0$. Since by Lemma 4.6 the matrices U_{st} and $U_{ii} + U_{st}$ belong to $\mathscr{A}'$ for all distinct s and t, and $T(\mathscr{A}') \subseteq \mathscr{A}'$, then the matrices $T(U_{st})$ and $T(U_{ii}) + T(U_{st})$ belong to $\mathscr{A}'$. By Lemma 4.6, all the matrices of $\mathscr{A}'$ have at most one nonzero principal element and two nonzero off-diagonal elements. Then we easily conclude that $T(U_{12}), T(U_{13}), T(U_{23})$ belong to $\langle U_{jj}, U_{rp} \rangle$, which contradicts the fact of T being nonsingular. Thus $h(T(U_{ii})) < 3$ for all i.

Similarly, if $i \neq k$, and if there are j, r, p with $r \neq p$ such that $(T(U_{ik}))_{jj} \neq 0$ and $(T(U_{ik}))_{rp} \neq 0$, we conclude that $T(U_{hh}) \in \langle U_{jj}, U_{rp} \rangle$ for all h, which cannot happen, because T is nonsingular. Thus $h(T(U_{ik})) < 3$ for all i and k.

Now if for all $i \in \{1, 2, 3\}$, we have $h(T(U_{ii})) = 1$, using the nonsingularity of T, we easily conclude that condition (i) holds.

CLAIM 2. *If $h(T(U_{ii})) = 2$ for some i, then $h(T(U_{rs})) = 1$ for all distinct r and s.*

Assume by contradiction that there are i, r, s, with $r \neq s$, such that $h(T(U_{ii})) = 2$ and $h(T(U_{rs})) \neq 1$. Then by Claim 1 we have that $h(T(U_{ii})) = 2$ and $h(T(U_{rs})) = 2$, which means that $T(U_{ii}) = aU_{kh}$ with $a \neq 0$ and $k \neq h$, and $T(U_{rs}) = a'U_{k'h'}$ with $a' \neq 0$ and $k' \neq h'$. As T is nonsingular, we have $\{k, h\} \neq \{k', h'\}$. Thus the matrix $T(U_{ii}) + T(U_{rs})$ has four nonzero elements, which cannot happen, because this matrix belongs to $\mathscr{A}'$.

So, using Claim 1, Claim 2, and the nonsingularity of T, we conclude that if we have $h(T(U_{ii})) = 2$ for some i, then condition (ii) holds. ∎

Proof of Theorem 2.3. Let T be a linear operator of $H_3(\mathbb{F})$ that preserves d_χ. Suppose there are $\pi, \tau \in S_3$ satisfying condition (i) of Lemma 4.7, and let $\sigma = \pi^{-1}$ and $\rho = \tau^{-1}$. Then there is $C \in H_3(\mathbb{F})$ such that the condition (2.3) is satisfied. Since T preserves d_χ, computing $d_\chi(T(I_3))$ and $d_\chi(T(U_{12} + U_{13} + U_{23}))$, we obtain (2.4).

Similarly, if T satisfies condition (ii) of Lemma 4.7, we obtain condition (b).

The converse is easily verified. ∎

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